Math 105 Chapter 7: More integration

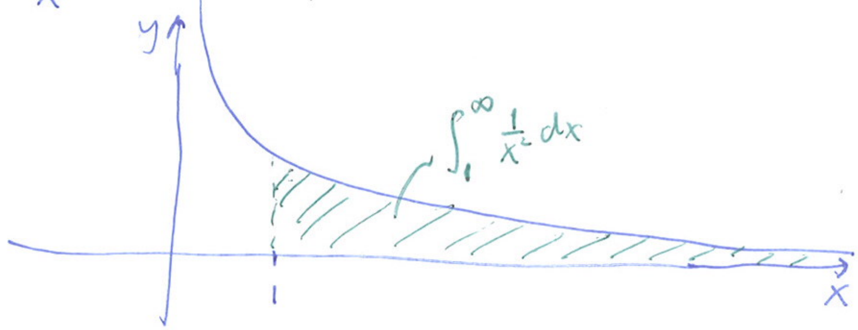
Improper Integrals

So far we have looked at the area under a continuous function on a closed interval [a, b]. Suppose a function is not defined on the end points, or the end points are infinite then we say

$$\int_a^b f(x)\,dx$$

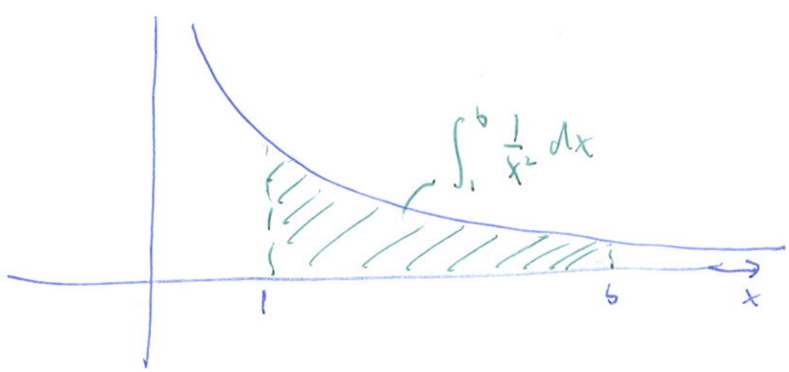
is an improper integral.

eg $f(x) = \frac{1}{x^2}$, what is the area under f from 1 to ∞.



We do the "obvious" thing and take limits.

We know the area between 1 and b for all b > 1.



So we let $b \to \infty$ and we get the area under f from 1 to infinity.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} 1 - \frac{1}{b}$$

$$= 1$$

Thus area under $\frac{1}{x^2}$ from 1 to infinity is 1.

Definition: Given $f(x)$ we define the following improper integrals.

① $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

② $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

③ $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ for all c.

If the above limits exist (and are independent of c in the case of ③), we say that the integral converges otherwise it diverges.

eg $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \log|x| \Big|_1^b$$

$$= \lim_{b\to\infty} \log b$$

$$= \infty$$

Thus $\int_1^\infty \frac{1}{x}\,dx$ diverges.

exercise: Show $\int_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p > 1$

diverges if and only if $0 < p \leq 1$.

eg $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{c} \frac{1}{1+x^2}\,dx + \int_c^\infty \frac{1}{1+x^2}\,dx$

$= \left[\arctan x\right]_{-\infty}^{c} + \left[\arctan x\right]_c^\infty$

$= \lim_{a \to \infty} \left[\arctan x\right]_a^c + \lim_{b \to \infty} \left[\arctan x\right]_c^b$

$= \lim_{a \to \infty} \arctan c - \arctan a + \lim_{b \to \infty} \arctan b - \arctan c$

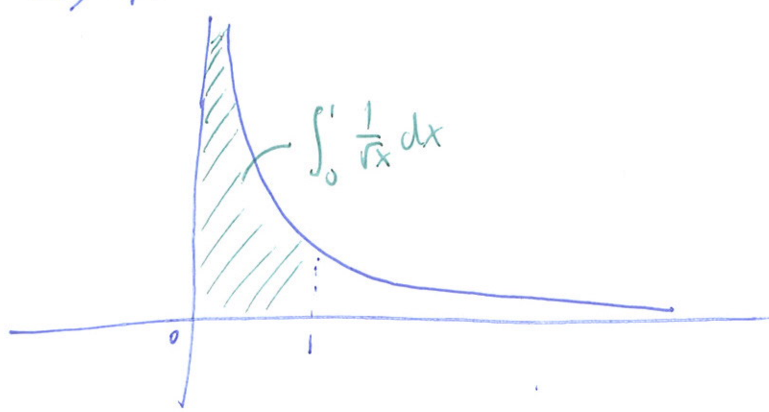
$= \lim_{b \to \infty} \arctan b - \lim_{a \to -\infty} \arctan a$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

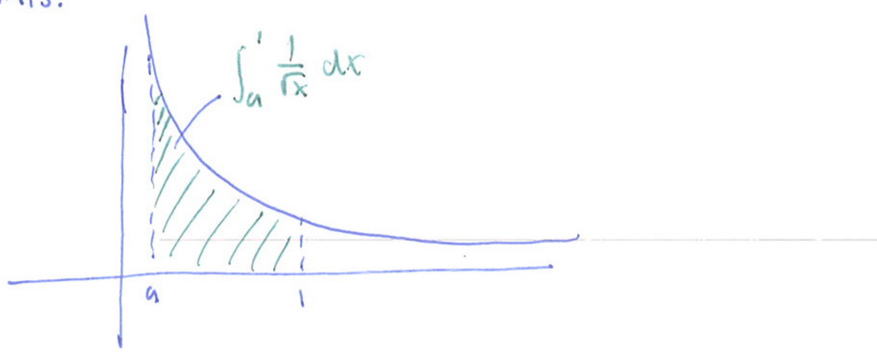
$= \pi$

Now what if we wanted to look at an integral of a function with a vertical asymptote?

eg $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.



We again take limits.



$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx$$, We look at 0^+ since we are only interested in what happens inside $[0,1]$.

$$= \lim_{a \to 0^+} \left[2\sqrt{x} \right]_a^1$$

$$= \lim_{a \to 0^+} 2 - 2\sqrt{a}$$

$$= 2$$

Definition. Given $f(x)$ we define the following improper integrals, on $[a,b]$.

1. If f is discontinuous at a, then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx$$

2. If f is discontinuous at b, then

$$\int_a^b f(x)\, dx = \lim_{d \to b^-} \int_a^d f(x)\, dx$$

3. If f is discontinuous at $p \in [a,b]$ then

$$\int_a^b f(x)\, dx = \lim_{d \to p^-} \int_a^d f(x)\, dx + \lim_{c \to p^+} \int_c^b f(x)\, dx$$

If the above limits exist, we say the integral converges, and diverges otherwise.

eg $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{d\to 0^-} \int_{-1}^{d} \frac{1}{x^2}\,dx$

$$= \lim_{d\to 0^-} \left[-\frac{1}{x} \right]_{-1}^{d}$$

$$= \lim_{d\to 0^-} 1 - \frac{1}{d}$$

$$= \infty$$

So $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\,dx$ diverges.

eg $\displaystyle\int_{0}^{\frac{\pi}{2}} \log(\sin(x)) \cos x\,dx$, $u = \sin x$, $du = \cos x\,dx$

$u(0) = 0$, $u(\frac{\pi}{2}) = 1$

$$= \int_0^1 \log u\,du \quad , \quad w = \log u \quad dv = du$$
$$dw = \frac{1}{u}\,du \quad v = u$$

$$= u\log u \Big|_0^1 - \int_0^1 1\,du$$

$$= \lim_{a\to 0^+} u\log u \Big|_a^1 - [u]_0^1 \quad , \text{ since } \log(0) \text{ does not exist}$$

$$= \lim_{a\to 0^+} (1\log 1 - a\log a) - (1-0)$$

$$= \lim_{a\to 0^+} -a\log a - 1$$

$$= \lim_{a\to 0^+} \frac{-\log a}{\frac{1}{a}} - 1 \quad , \frac{\infty}{\infty} \text{ so apply l'hopital}$$

$$= \lim_{a\to 0^+} \frac{-\frac{1}{a}}{-\frac{1}{a^2}} - 1$$

$$= \lim_{a\to 0^+} a - 1$$

Thus $\int_0^{\pi/2} \log(\sin x)\cos x\,dx = -1$

eg $\int_{-1}^{1} \frac{1}{x}\,dx$

$= \lim_{d\to 0^-} \int_{-1}^{d} \frac{1}{x}\,dx + \lim_{c\to 0^+} \int_{c}^{1} \frac{1}{x}\,dx$

$= \lim_{d\to 0^-} \Big[\log|x|\Big]_{-1}^{d} + \lim_{c\to 0^+} \Big[\log|x|\Big]_{c}^{1}$

$= \lim_{d\to 0^-} \log|d| - \log|-1| + \lim_{c\to 0^+} \log|1| - \log|c|$

$= \lim_{d\to 0^-} \log|d| - \lim_{c\to 0^+} \log|c|$

$= -\infty - (+\infty)$

$= -\infty + \infty$

Limit does not exist and thus $\int_{-1}^{1} \frac{1}{x}\,dx$ diverges.

Integral approximations:

Recall the reason why we spent so much time on antiderivatives was so that we could find

$$\int_a^b f(x)\,dx.$$

But what if $f(x) = e^{x^2}$, $\sin(x^2)$, $\sqrt{1+x^3}$, etc, where no nice closed form of an antiderivitive exists. These functions come up in practice and we often need to approximate the integral.

Recall a Reimann sums we used to approximate the integral using rectangles.

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x \xrightarrow[n\to\infty]{} \int_a^b f(x)\,dx$$

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

Right endpoint $\Rightarrow$ $x_i^* = x_i = a + i\Delta x$

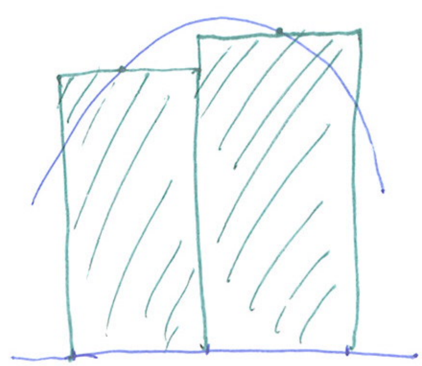
Left endpoint $\Rightarrow$ $x_i^* = x_{i-1} = a + (i-1)\Delta x$

Midpoint rule $\Rightarrow$ $x_i^* = \dfrac{x_i + x_{i-1}}{2} = a + (i-\tfrac{1}{2})\Delta x$

Although all these are approximations which is the best one?

right endpoint

left endpoint



midpoint

- right endpoint <u>overshoots</u> when f is <u>increasing</u>
  <u>undershoots</u> when f is <u>decreasing</u>

- left endpoint <u>undershoots</u> when f is <u>increasing</u>
  <u>overshoots</u> when f is <u>decreasing</u>

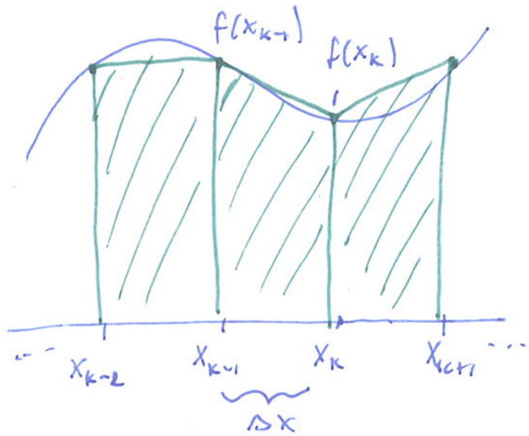
- Midpoint rule overshoots sometimes and undershoots other times averaging out to a good approximation.

So let $M(n) = \Delta x \left(\sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2} \right) \right) \approx \int_a^b f(x)dx$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

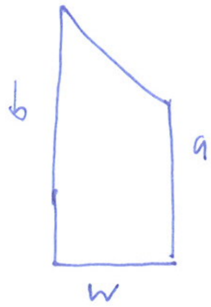
$M(n)$ is called the <u>midpoint rule</u> and is just the midpoint Reimann Sum.

Now midpoint rule was the best approximation using rectangles. But what if we used other shapes to approximate, for example a trapezoid.



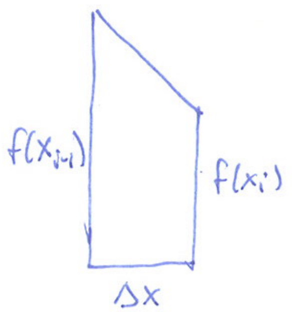
We again break up our interval $[a, b]$ into n equal parts, but instead of n rectangles we have n trapezoids.

Recall the area of the trapezoid



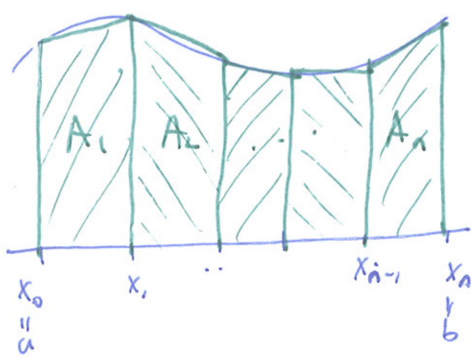
$$\Rightarrow A = \left(\frac{b+a}{2}\right) w$$

So if A_i is the area of the i^{th} trapezoid then



$$\Rightarrow A_i = \left(\frac{f(x_{i-1}) + f(x_i)}{2}\right) \Delta x$$

So let $T(n)$ be the sum of the n trapezoids



$$T(n) = A_1 + A_2 + A_3 + \cdots + A_n$$

$$= \frac{f(x_0) + f(x_1)}{2}\Delta x + \frac{f(x_1) + f(x_2)}{2}\Delta x + \frac{f(x_2) + f(x_3)}{2}\Delta x + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2}\Delta x$$

$$= \Delta x \left(\frac{f(x_0)}{2} + \underbrace{\frac{f(x_1)}{2} + \frac{f(x_1)}{2}}_{f(x_1)} + \underbrace{\frac{f(x_2)}{2} + \frac{f(x_2)}{2}}_{f(x_2)} + \underbrace{\frac{f(x_3)}{2} + \cdots}_{f(x_3)} + \underbrace{\frac{f(x_{n-1})}{2} + \frac{f(x_{n-1})}{2}}_{f(x_{n-1})} + \frac{f(x_n)}{2} \right)$$

Thus we have

$$T(n) = \Delta x \left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2} \right)$$

$T(n)$ is called the Trapezoid rule.

There is an other one called Simpson's rule that shows up by approximating the integral by parabola's. It is much more complicated so wont derive it.

$$S(n) = \frac{\Delta x}{3} \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n) \right)$$

if n is even.

That is a complicated formula, but just remember the coefficients start and end with 1 and alternate with 4's and 2's in the middle.

$$(1, 4, 2, 4, 2, 4, 2, \ldots, 4, 2, 4, 1).$$

Now that we have these approximations we want to quantify how good the approximations are to the real value. We call this error.

Definition: The absolute error and relative error are given by:

$$\text{Absolute error} = |(\text{true value}) - (\text{approximation})|$$

$$\text{Relative error} = \frac{|\text{true value} - \text{approximation}|}{|\text{true value}|}$$

Where we are assuming $|\text{true value}| \neq 0$ in the second formula.

eg Suppose I have a jar of 7153 jelly beans. You guess that there are 7000, then the absolute and relative error are:

$$\text{Absolute error} = |7153 - 7000|$$
$$= 153$$
$$\text{relative error} = \frac{|7153 - 7000|}{7153}$$
$$\approx 0.02138$$

Relative error takes into account the size of your actual measurement. Looking at an extreme example suppose there were only 97 beans and you guessed there we 250, then

$$\text{Absolute error} = |97 - 250|$$
$$= 153$$
$$\text{Relative error} = \frac{|97 - 250|}{97}$$
$$= 1.577$$

So the absolute error is the same in both cases but the relative is much higher. Clearly the first guess was better.

In general we want to minimize the error. In the case of our integral approximations we don't always know the actual value so we can't compute error explicitely. But we can find an upper bound on what the error should be.

Theorem: If f'' is continous on $[a,b]$ and there is a M such that $|f''(x)| \leq M$ for all $x \in [a,b]$, then

$$① \quad \left| \int_a^b f(x)\,dx - M(n) \right| \leq \frac{M(b-a)}{24}(\Delta x)^2 = \frac{M(b-a)^3}{24n^2}$$

$$② \quad \left| \int_a^b f(x)\,dx - T(n) \right| \leq \frac{M(b-a)}{12}(\Delta x)^2 = \frac{M(b-a)^3}{12n^2}$$

If $f^{(4)}$ is cts on $[a,b]$ and there is a K such that $|f^{(4)}(x)| \leq K$ for all $x \in [a,b]$, then

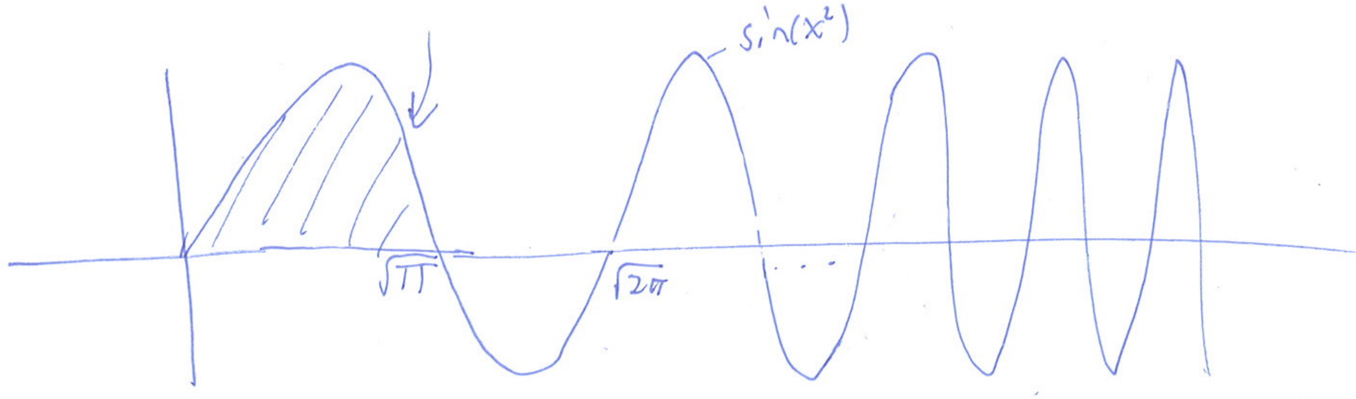
$$③ \quad \left| \int_a^b f(x)\,dx - S(n) \right| \leq \frac{K(b-a)}{180}(\Delta x)^4 = \frac{K(b-a)^5}{180n^4}$$

So the above theorem tells us that the absolute error of $\int_a^b f(x)dx$ when approximating with $M(n)$ is less than the absolute error of $\int_a^b f(x)dx$ when approximating with $T(n)$. Thus $M(n)$ is a better approximation than $T(n)$!

③ tells us that Simpson's rule blows the other 2 away.

Also as n gets large the absolute error goes to 0, as expected.

eg Approximate $\int_0^{\sqrt{\pi}} \sin(x^2)\,dx$ using $n=4$



Note $\int \sin(x^2)\,dx$ cannot be expressed in terms of elementary functions.

So we must approximate

$$\Delta x = \frac{b-a}{n} = \frac{\sqrt{\pi}-0}{4} = \frac{\sqrt{\pi}}{4}$$

$$x_i = a + i\Delta x = i\frac{\sqrt{\pi}}{4} \implies x_0 = 0,\ x_1 = \frac{\sqrt{\pi}}{4},\ x_2 = \frac{\sqrt{\pi}}{2},\ x_3 = 3\frac{\sqrt{\pi}}{4},\ x_4 = \sqrt{\pi}$$

$$M(4) = \Delta x \sum_{i=1}^{n} f(a + (i - \tfrac{1}{2})\Delta x)$$

$$= \frac{\sqrt{\pi}}{4} \sum_{i=1}^{4} \sin\left[\left((i-\tfrac{1}{2})\frac{\sqrt{\pi}}{4}\right)^2\right]$$

$$= \frac{\sqrt{\pi}}{4}\left[\sin\left(\left(\frac{1}{2}\frac{\sqrt{\pi}}{4}\right)^2\right) + \sin\left(\left(\frac{3}{2}\frac{\sqrt{\pi}}{4}\right)^2\right) + \sin\left(\left(\frac{5}{2}\frac{\sqrt{\pi}}{4}\right)^2\right) + \sin\left(\left(\frac{7}{2}\frac{\sqrt{\pi}}{4}\right)^2\right)\right]$$

$$= \frac{\sqrt{\pi}}{4}\left[\sin\left(\frac{\pi}{64}\right) + \sin\left(\frac{9\pi}{64}\right) + \sin\left(\frac{25\pi}{64}\right) + \sin\left(\frac{49\pi}{64}\right)\right]$$

$$= 0.925986\ldots$$

$$T(4) = \Delta x \left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + f(x_3) + \frac{f(x_4)}{2} \right]$$

$$= \frac{\sqrt{\pi}}{4} \left[\frac{1}{2}\sin(0^2) + \sin\left(\left(\frac{\sqrt{\pi}}{4}\right)^2\right) + \sin\left(\left(\frac{\sqrt{\pi}}{2}\right)^2\right) + \sin\left(\left(\frac{3\sqrt{\pi}}{4}\right)^2\right) + \frac{1}{2}\sin\pi \right]$$

$$= \frac{\sqrt{\pi}}{4} \left[\sin\left(\frac{\pi}{16}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{9\pi}{16}\right) \right]$$

$$= 0.834375\ldots$$

$$S(4) = \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right]$$

$$= \frac{\sqrt{\pi}}{4} \left[\sin(0) + 4\sin\left(\frac{\pi}{16}\right) + 2\sin\left(\frac{\pi}{4}\right) + 4\sin\left(\frac{9\pi}{16}\right) + \sin(\pi) \right]$$

$$= 0.895449\ldots$$

Now lets see how these approximations are. To do this lets compute derivatives.

$f(x) = \sin x^2$

$f'(x) = 2x\cos x^2$

$f''(x) = 2\cos x^2 - 4x^2 \sin x^2$

$f'''(x) = -12x\sin x^2 - 8x^3\cos x^2$

$f^{(4)}(x) = -12x\sin x^2 - 48x^2\cos(x^2) + 16x^4\sin(x^4)$

We want a bound of $f^{(2)}$ and $f^{(4)}$ on $[0, \sqrt{\pi}]$.

$$|f''(x)| = |2\cos^2 x - 4x^2 \sin x^2|$$

$$\leq 2|\cos^2 x| + 4x^2 |\sin x^2 \quad \text{, by triangle inequality}$$

$$\leq 2 + 4x^2 \quad \text{, since } |\cos x^2|, |\sin x^2| \leq 1$$

$$\leq 2 + 4\pi \quad \text{, since } x \in [0, \sqrt{\pi}]$$

$$|f^{(4)}(x)| = |-12x\sin x^2 - 48x^2\cos x^2 + 16x^4 \sin x^2|$$

$$\leq |12x\sin x^2| + |48x^2\cos x^2| + |16x^4\sin x^2| \quad \text{, by triangle inequality } |a+b| \leq |a|+|b|$$

$$= 12|x||\sin x^2| + 48|x|^2|\cos x^2| + 16|x|^4|\sin x^2|$$

$$\leq 12|x| + 48|x|^2 + 16|x|^4 \quad \text{, since } |\sin x^2|, |\cos x^2| \leq 1$$

$$\leq 12\sqrt{\pi} + 48(\sqrt{\pi})^2 + 16(\sqrt{\pi})^4$$

$$= 12\sqrt{\pi} + 48\pi + 16\pi^2$$

So with the notation the approximation theorem,

$$M = 2 + 4\pi$$

$$K = 12\sqrt{\pi} + 48\pi + 16\pi^2$$

$$\left|\int_0^{\sqrt{\pi}} \sin x^2\,dx - M(4)\right| \leq \frac{M(b-a)^3}{24n^2} = \frac{M(\sqrt{\pi}-0)^3}{24 \cdot 4^2} \approx 0.211\ldots$$

$$\left|\int_0^{\sqrt{\pi}} \sin x^2\,dx - T(4)\right| \leq \frac{M(b-a)^3}{12n^2} = \frac{M(\sqrt{\pi}-0)^3}{12 \cdot 4^2} \approx 0.422\ldots$$

$$\left|\int_0^{\sqrt{\pi}} \sin x^2\,dx - S(4)\right| \leq \frac{K(b-a)^5}{180n^4} = \frac{(12\sqrt{\pi} + 48\pi + 16\pi^2)(\sqrt{\pi}-0)^5}{180 \cdot 4^4} \approx 0.125\ldots$$

So $M(4)$, $T(4)$ are terrible approximations.

So the absolute error using simpson's rule is the smallest, implying $S(4)$ is the best approximation of the 3.

Indeed $\int_0^{\sqrt{\pi}} \sin x^2 \, dx \approx 0.89483..$

So the relative error is at most

Midpoint rule: $\dfrac{\left| \int_0^{\sqrt{\pi}} \sin x^2 dx - M(4) \right|}{\left| \int_0^{\sqrt{\pi}} \sin x^2 dx \right|} \approx \dfrac{0.211...}{0.89483} \approx 0.236...$

Trapezoid rule: $\dfrac{\left| \int_0^{\sqrt{\pi}} \sin x^2 dx - T(4) \right|}{\left| \int_0^{\sqrt{\pi}} \sin x^2 dx \right|} \approx \dfrac{0.422...}{0.89483} \approx 0.472...$

Simpson's rule: $\dfrac{\left| \int_0^{\sqrt{\pi}} \sin x^2 dx - S(4) \right|}{\int_0^{\sqrt{\pi}} \sin x^2 dx} \approx \dfrac{0.125...}{0.89483} \approx 0.1400...$

So simpson's rule is within 14% of the actual answer.